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Path-Integral Quantization and the Ground-State Functional for Maxwell–Chern–Simons System

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The Maxwell–Chern–Simons system as a constrained system is quantized in the path integral formulation. Using the functional partition function and the method proposed by Fradkin, we obtain the correct absolute value squared of the ground state.

Quantum field theories in (2 + 1)-dimensional space-time, especially Chern–Simons systems, have become the focus of widespread research activity (Deser *et al.*, 1982), not only for pedagogical and mathematical reasons, but also because of their possible role in (2 + 1)-dimensional condensed-matter physics (Hall effect, high T_c). These models are particularly interesting since they possess special topological structures which are only available in odd-dimensional space-time. The Maxwell–Chern–Simons system has been quantized canonically, and as an exactly solvable model, the ground-state functional has also been obtained (Deser *et al.*, 1982). Recently, Fradkin (1993) proposed a marvelous method for the calculation of state functionals in the Schrödinger representation. This method has been used to calculate ground-state functionals of a number of models, e.g., the Thirring–Luttinger model, coset models, and the Sutherland model (Fradkin and Moreno, 1993). In particular, it was shown that the wave functionals of the liquid ground states of fractional quantum Hall systems, in the thermodynamic limit, are universal at long distances and that they have a generalized Laughlin form (Lopez and Fradkin, 1992). All these systems are nonsingular. In this paper, we apply this path-integral method to the Maxwell–Chern–Simons system which is singular, and obtain the correct absolute value of the ground state.

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The Lagrangian density of the model considered is (Deser *et al.*, 1982)

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4} \epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

We use the signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The equations of motion

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{3}$$

are invariant against the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Omega(x) \tag{4}$$

while the Lagrangian changes by a total derivative

$$\mathcal{L} \to \mathcal{L} + \partial_\alpha\left(\frac{\mu}{4} \epsilon^{\alpha\mu\nu}F_{\mu\nu}\Omega\right) \tag{5}$$

In order to quantize (canonically or via path integral) the system, it is necessary to analyze the classical canonical structure because quantization needs true physical contents. According to the definition, the canonical momentum conjugate to A_μ is

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{i0} + \frac{\mu}{2} \epsilon^{ij}A_j \tag{6}$$

$$\pi^0 = 0 \tag{7}$$

The nonvanishing Poisson brackets are

$$\{\pi^i(x), A_j(y)\}_{x^0=y^0} = -\delta^i_j\delta^2(\mathbf{x} - \mathbf{y}) \tag{8}$$

Thus we have a primary constraint

$$\phi_1 = \pi^0 \approx 0 \tag{9}$$

The canonical Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_c &= \pi^i\dot{A}_i - \mathcal{L} \\ &= -\frac{1}{2} \pi^i\pi_i + \frac{\mu}{2} \epsilon^{ij}\pi_iA_j - \frac{\mu^2}{8} A^iA_i + \frac{1}{4} F^{ij}F_{ij} - \frac{\mu}{4} \epsilon^{ij}F_{ij}A_0 + \pi^i\partial_iA_0\end{aligned} \tag{10}$$

The total Hamiltonian is thus

$$H_T = \int d^2\mathbf{x}\,(\mathcal{H}_c + \alpha\phi_1) \tag{11}$$

where $\alpha(x)$ is the Lagrangian multiplier. From the consistency condition of the primary constraint

$$\{\phi_1, H_T\} \approx 0 \tag{12}$$

we obtain the secondary constraint

$$\phi_2 = \partial_i \pi^i + \frac{\mu}{4}\,\epsilon^{ij} F_{ij} \tag{13}$$

It can be easily shown that

$$\{\phi_2, H_T\} \approx 0 \tag{14}$$

Thus there exist no further constraints. Hence we have two constraints which are first class

$$\{\phi_1, \phi_2\} \approx 0 \tag{15}$$

It is well known that, according to Dirac's conjecture, the first-class constraints generate gauge transformations; so, to eliminate the nondynamical freedom of a singular system which has n first-class constraints, n gauge-fixing conditions must be chosen. Hence here the true phase space is of dimension $2 \times 3 - 2 \times 2 = 2$. We choose the familiar Coulomb gauge

$$f_2 = \partial_i A^i \approx 0 \tag{16}$$

The consistent condition $\dot{f}_2 \approx 0$ and the definition of π^i in (6) imply that another gauge condition may be chosen to be

$$f_1 = \partial_i \pi^i + \nabla^2 A^0 - \frac{\mu}{2}\,\epsilon^{ij}\partial_i A_j \approx 0 \tag{17}$$

It can be shown that the determinant $\det\{\phi_a, \phi_b\} \neq 0$ and is independent of the field variables, hence we have the path-integral partition function according to the general scheme (Li, 1993; Gitman and Tyutin, 1990)

$$Z[J] = \int \mathcal{D}\pi^\mu\, \mathcal{D}A_\mu\, \delta(\pi^0)\delta\left(\partial_i\pi^i + \frac{\mu}{4}\,\epsilon^{ij}F_{ij}\right)\delta(\partial_i A^i)$$
$$\times\,\delta\left(\partial_i\pi^i + \nabla^2 A^0 - \frac{\mu}{2}\,\epsilon^{ij}\partial_i A_j\right)$$

$$\times \exp\left\{i \int d^3x \, [\pi^\mu \dot{A}_\mu - \mathcal{H}_c + A_\mu J^\mu]\right\} \tag{18}$$

The delta function $\delta(\pi^0)$ enables us to integrate π^0 first, i.e.,

$$Z[J] = \int \prod_{i,\mu} \mathcal{D}\pi^i \, \mathcal{D}A_\mu \, \delta\left(\partial_i \pi^i + \frac{\mu}{4} \epsilon^{ij} F_{ij}\right)$$

$$\times \delta(\partial_i A^i) \delta\left(\partial_i \pi^i + \nabla^2 A^0 - \frac{\mu}{2} \epsilon^{ij} \partial_i A_j\right)$$

$$\times \exp\left\{i \int d^3x \, [\pi^i \dot{A}_i - \mathcal{H}_c(\pi^i, A_\mu, \partial_i A_\mu) + A_\mu J^\mu]\right\} \tag{19}$$

Making use of a variable transformation which does not change the partition function

$$\pi^i \to \pi^i - \frac{\mu}{2} \epsilon^{ij} A_j \tag{20}$$

$$\prod_{i,\mu} \mathcal{D}\pi^i \, \mathcal{D}A_\mu \to \text{const} \cdot \prod_{i,\mu} \mathcal{D}\pi^i \, \mathcal{D}A_\mu \tag{21}$$

we have

$$Z[J] = \int \prod_{i,\mu} \mathcal{D}\pi^i \, \mathcal{D}A_\mu \, \delta(\partial_i \pi^i) \delta(\partial_i A^i) \delta(\nabla^2 A^0 - \mu \epsilon^{ij} \partial_i A_j)$$

$$\times \exp\left\{i \int d^3x \left[\pi^i \dot{A}_i - \frac{\mu}{2} \epsilon^{ij} \dot{A}_i A_j + \frac{1}{2} \pi^i \pi_i + \frac{\mu^2}{2} A^i A_i \right.\right.$$

$$\left.\left. - \frac{1}{4} F^{ij} F_{ij} + J^\mu A_\mu \right]\right\} \tag{22}$$

The delta function $\delta(\nabla^2 A^0 - \mu \epsilon^{ij} \partial_i A_j)$ ensures that

$$\partial^i A^0 \approx -\mu \epsilon^{ij} A_j \tag{23}$$

so A_0 can be integrated. Notice that

$$-\frac{\mu}{2} \epsilon^{ij} \dot{A}_i A_j \approx \frac{1}{2} \dot{A}_i \partial^i A^0 \approx \frac{1}{2} \partial^i (\dot{A}_i A^0) \tag{24}$$

i.e., the second term in the exponential is weakly a surface term, which can be neglected. Hence we have

$$Z[J] = \int \prod_i \mathcal{D}\pi^i \, \mathcal{D}A_i \, \delta(\partial_i \pi^i)\delta(\partial_i A^i)$$
$$\times \exp\left\{ i \int d^3x \left[\pi^i \dot{A}_i + \frac{1}{2}\pi^i \pi_i + \frac{\mu^2}{2} A^i A_i \right.\right.$$
$$\left.\left. - \frac{1}{4} F^{ij} F_{ij} + J^i A_i - J^0 \frac{\mu}{\partial^i} \epsilon_{ij} A^j \right]\right\} \tag{25}$$

Due to the delta functions, only the transverse parts of the fields contribute to the integral. Consider the external source $J^\mu = (0, \mathbf{J})$; we have

$$Z[\mathbf{J}_T] = \int \prod_i \mathcal{D}\pi^i_T \, \mathcal{D}A_{Ti} \exp\left\{ i \int d^3x \left[\pi^i_T \dot{A}_{Ti} + \frac{1}{2} \pi^i_T \pi_{Ti} \right.\right.$$
$$\left.\left. + \frac{\mu^2}{2} A^i_T A_{Ti} - \frac{1}{4} F^{ij} F_{ij} + J^i_T A_{Ti} \right]\right\} \tag{26}$$

where $\mathbf{J}_T$ is the transverse part of $\mathbf{J}$. This partition function is in agreement with the canonical Hamiltonian obtained in Deser *et al.* (1982). After integrating $\boldsymbol{\pi}$, we have finally

$$Z[\mathbf{J}_T] = \int \mathcal{D}\mathbf{A}_T \exp\left\{ i \int d^3x \left[\frac{1}{2} \dot{\mathbf{A}} + \frac{1}{2} \mathbf{A}_T \cdot (\nabla^2 - \mu^2) \mathbf{A}_T + \mathbf{J}_T \cdot \mathbf{A}_T \right]\right\} \tag{27}$$

We next evaluate the absolute value of the ground state by means of this $Z[\mathbf{J}_T]$.

Quite recently, Fradkin (1993) obtained a relationship between the ground-state functional and the path-integral partition function. For a scalar field ϕ, the absolute value of the ground state $\Psi_{gs}[\phi(\mathbf{x})]$ is

$$|\Psi_{gs}[\phi(\mathbf{x})]|^2 = \int \mathcal{D}\mathbf{J}(\mathbf{x}) \exp\left\{ -i \int d\mathbf{x} \, \mathbf{J}(\mathbf{x})\phi(\mathbf{x}) \right\} Z[\mathbf{J}]_{t_0} \tag{28}$$

where $Z[\mathbf{J}]_{t_0}$ comes from the restriction of the source $\mathbf{J}(x)$ which appears in the path integral to the form $\mathbf{J}(x) = \mathbf{J}(\mathbf{x})\delta(t - t_0)$. In the present case

$$|\Psi_{gs}[\mathbf{A}_T]|^2 = \int \mathcal{D}\mathbf{J}_T(\mathbf{x}) \exp\left\{ -i \int d^2\mathbf{x} \, \mathbf{J}_T(\mathbf{x}) \cdot \mathbf{A}_T(\mathbf{x}) \right\} Z[\mathbf{J}_T]_{t_0} \tag{29}$$

The partition function can be evaluated from (27), i.e.,

$$Z[\mathbf{J}_T] = \mathcal{N} \exp\left\{ -\frac{i}{2} \int d^3x \, d^3y \, J^i(x) G_{ij}(x, y) J^j(y) \right\} \tag{30}$$

where the Green function is given by

$$G_{ij}(x, y) = \frac{\delta_{ij}}{(2\pi)^3} \int \frac{e^{-ik(x-y)}}{k_0^2 - (\mathbf{k}^2 + \mu^2) + i\eta} d^3k \tag{31}$$

where $\eta > 0$, so

$$G_{ij}(\mathbf{x}, \mathbf{y}) = \lim_{x^0 \to y^0} G_{ij}(x, y) = \frac{\delta_{ij}}{(2\pi)^2} (-i) \int \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{2\omega(\mathbf{k})} d^2\mathbf{k} \tag{32}$$

where $\omega(\mathbf{k}) = (\mathbf{k}^2 + \mu^2)^{1/2}$. Hence

$$Z[\mathbf{J}]_{t_0} = \mathcal{N} \exp\left\{ -\frac{1}{2} \int d^2\mathbf{x}\, d^2\mathbf{y}\, J^i(\mathbf{x}) G^e_{ij}(\mathbf{x}, \mathbf{y}) J^j(\mathbf{y}) \right\} \tag{33}$$

where

$$G^e_{ij}(\mathbf{x}, \mathbf{y}) \equiv i G_{ij}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{ij}}{(2\pi)^2} \int \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{2\omega(\mathbf{k})} d^2\mathbf{k} \tag{34}$$

Thus, from (27), (29), and (33), we have the absolute value squared of the ground-state functional

$$\begin{aligned} &|\Psi_{gs}[\mathbf{A}_T]|^2 \\ &= \mathcal{N} \exp\left\{ -\int d^2\mathbf{x}\, d^2\mathbf{y}\, \mathbf{A}_T(\mathbf{x}) \cdot \mathbf{A}_T(\mathbf{y}) \int \omega(\mathbf{k}) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} d^2\mathbf{k} \right\} \\ &= \mathcal{N} \exp\left[-\int d^2\mathbf{x}\, \mathbf{A}_T \cdot (-\nabla^2 + \mu^2)^{1/2} \mathbf{A}_T \right] \end{aligned} \tag{35}$$

which is correct (Deser *et al.*, 1982).

We make final some remarks. It can be seen from the partition function (27) that the model (1) is in fact a massive gauge theory. On the other hand, since the transverse part $\mathbf{A}_T$ has only one freedom, the property of the system must be analogous to that of a scalar field with mass μ. Their relationship was also given explicitly in Deser *et al.* (1982). It can be understood from this paper that Fradkin's approach to the evaluation of wave functionals can also be applied to singular systems without special difficulty.

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